

On the (2+1)-dimensional Dirac equation in a constant magnetic field with a minimal length uncertainty

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Abstract

We exactly solve the (2+1)-dimensional Dirac equation in a constant magnetic field in the presence of a minimal length. Using a proper ansatz for the wave function, we transform the Dirac Hamiltonian into two 2-dimensional non-relativistic harmonic oscillator and obtain the solutions without directly solving the corresponding differential equations which are presented by Menculini et al. [Phys. Rev. D 87, 065017 (2013)]. We also show that Menculini et al. solution is a subset of the general solution which is related to the even quantum numbers.

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1 Introduction

At high energy limit, close to the Planck scale where the corresponding Schwarzschild radius becomes comparable with the Compton wavelength and both tend to the Planck length, the effects of gravity become so important that would result in discreteness of the spacetime. In this case, different approaches to quantum gravity such as string theory [1–5], noncommutative geometry [6], and loop quantum gravity [7] predict the existence of a minimal measurable length. These theories argue that near the Planck scale, the Heisenberg Uncertainty Principle should be replaced by the so-called Generalized Uncertainty Principle (GUP).

Recently, various studies about the effects of the minimal length have been done in the literature such as hydrogen atom spectrum [8–11], Lamb shift [9, 10, 12], harmonic oscillator [13–18], gravitational quantum well [19], Casimir effect [20, 21], particles scattering [22–24], resolution of wave function singularities

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for strongly attractive potentials [25], and the classical limit of the minimal length uncertainty [26]. Also, the effects of the GUP on the electromagnetic field is discussed in Ref. [27].

The effects of the minimal length and maximal momentum on relativistic and non-relativistic wave equations have been investigated in Refs. [28–31]. The problems of Lorentz violation, Dirac particle in a box, Dirac oscillator [28], potential step, and potential barrier [29] are studied in the presence of the minimal length uncertainty. Using a recent proposal by Ali, Das and Vagenas that implies a minimal length uncertainty and a maximal momentum [31], the problems of superconductivity and the quantum Hall effect are discussed in Ref. [30]. Moreover, the solutions of the 3-dimensional Dirac oscillator in the presence of the minimal length are presented Ref. [15].

The solutions of the Dirac equation in the presence of a homogeneous magnetic field have many applications in condensed matter physics. Recently, for the 2-dimensional electron systems, *i.e.* non-relativistic Landau levels, the expected ring like internal structure of the wave functions is detected for the first time using scanning tunneling spectroscopy [32]. Also, the massless (2+1)-dimensional Dirac equation is used to describe the motion of electrons in graphene [33] and to find an upper bound on the fundamental minimal length $\hbar\sqrt{\beta}$ using experimental measurements of the relativistic Landau levels in graphene [34]. Notice that, if we take β as a non-universal parameter, this upper bound could be varied from one experiment to another. For instance, in the Lamb shift an upper bound for the minimal length is obtained $\hbar\sqrt{\beta} < 10^{-17}m$ which is of the order of the electroweak scale [11, 26]. But, for the ultracold neutron energy levels in a gravitational quantum well [19, 35], the upper bound is $\hbar\sqrt{\beta} < 2.41 \times 10^{-9}m$ which agrees well with the results that obtained for the massless (2+1)-dimensional Dirac equation [36].

In this paper, we study the problem of the (2+1)-dimensional Dirac equation in the presence of a constant magnetic field and a minimal length uncertainty. This problem is recently investigated in Refs. [36, 37] by directly solving the corresponding differential equations in momentum space. However, as we shall show, using a proper ansatz for the momentum space wave function, the Dirac Hamiltonian can be cast into two 2-dimensional non-relativistic harmonic oscillators which is exactly solvable in terms of Jacobi polynomials [16]. Then, without directly solving the differential equations, we find exact energy

eigenvalues and eigenfunctions and show that Menculini *et al.* results correspond to the even quantum numbers as a part of the general solution.

2 The generalized uncertainty principle

Let us consider the generalized uncertainty principle in the form [17]

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2} \delta_{ij} \left(1 + \beta \left((\Delta P)^2 + \langle P \rangle^2 \right) \right), \quad (1)$$

where β is the GUP parameter. The above inequality relation leads to the existence of a minimal measurable length $(\Delta X)_{min} = \hbar\sqrt{\beta}$ which is of the order of the Planck length $\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\text{m}$ [17, 18, 38]. This uncertainty relation leads to the following deformed commutation relation, namely

$$[X_i, P_j] = i\hbar\delta_{ij} (1 + \beta P^2), \quad (2)$$

where $P^2 = \sum_i P_i^2$. It is straightforward to check that when $\beta = 0$, the well-known commutation relation in ordinary quantum mechanics is recovered. In momentum space representation, we have

$$P_i \psi(p) = p_i \psi(p), \quad (3)$$

$$X_i \psi(p) = i\hbar(1 + \beta P^2) \partial_{p_i} \psi(p). \quad (4)$$

Now, using Eq. (4), the commutation relations for position operators reads

$$[X_i, X_j] = 2i\hbar\beta(P_i X_j - P_j X_i), \quad (5)$$

as a noncommutative generalization of the position space. Note that, the rotational symmetry is not broken by the commutation relations (2) and (5). In fact, we can still express the generators of rotations in terms of the position and momentum operators as

$$L_{ij} = \frac{1}{1 + \beta \vec{P}^2} (X_i P_j - X_j P_i), \quad (6)$$

where in the limit $\beta \rightarrow 0$, we obtain the ordinary definition of orbital angular momentum.

3 Dirac equation in the GUP framework

The Dirac Hamiltonian in the presence of a homogeneous magnetic field $\vec{B} = (0, 0, B_0)$ takes the form

$$H = c\vec{\sigma} \cdot \left(\vec{P} + \frac{e}{c}\vec{A} \right) + \sigma_z Mc^2, \quad (7)$$

where, $\vec{\sigma}$ and \vec{A} denote the Pauli matrices and the vector potential in the symmetric gauge, respectively

$$A_x = -\frac{1}{2}B_0Y, \quad A_y = \frac{1}{2}B_0X. \quad (8)$$

The eigenvalue problem is

$$H\psi(p) = E\psi(p), \quad \psi(p) = \begin{pmatrix} \psi^{(1)}(p) \\ \psi^{(2)}(p) \end{pmatrix}. \quad (9)$$

If we define P_{\pm} as

$$P_{\pm} = P_x \pm iP_y = \left(P_x + \frac{e}{c}A_x \right) \pm i \left(P_y + \frac{e}{c}A_y \right), \quad (10)$$

the eigenvalue equation can be written as

$$H\psi(p) = \begin{pmatrix} Mc^2 & cP_- \\ cP_+ & -Mc^2 \end{pmatrix} \begin{pmatrix} \psi^{(1)}(p) \\ \psi^{(2)}(p) \end{pmatrix} = E \begin{pmatrix} \psi^{(1)}(p) \\ \psi^{(2)}(p) \end{pmatrix}. \quad (11)$$

Thus, we have

$$P_- \psi^{(2)}(p) = \epsilon_- \psi^{(1)}(p), \quad P_+ \psi^{(1)}(p) = \epsilon_+ \psi^{(2)}(p), \quad (12)$$

where $\epsilon_{\pm} = (E \pm Mc^2)/c$. By separating the components of Eq. (11) and using Eq. (12) we find

$$H^{(1)}\psi^{(1)}(p) = P_- P_+ \psi^{(1)}(p) = \epsilon^2 \psi^{(1)}(p), \quad (13)$$

$$H^{(2)}\psi^{(2)}(p) = P_+ P_- \psi^{(2)}(p) = \epsilon^2 \psi^{(2)}(p), \quad (14)$$

where $\epsilon^2 = \epsilon_+ \epsilon_- = (E^2 - M^2 c^4)/c^2$.

Before proceed further, let us consider the problem of the non-relativistic harmonic oscillator in two dimensions which is exactly solvable [16]. Its Hamiltonian is given by

$$H = \frac{1}{2\mu}P^2 + \frac{1}{2}\mu\omega^2(X^2 + Y^2), \quad (15)$$

where the eigenvalue equation is given by $H\psi(p) = E\psi(p)$. Since this Hamiltonian is rotationally symmetric, the energy eigenfunction can be written as a product of a radial wave function and spherical harmonics. So, we have

$$\psi(p) = \frac{1}{\sqrt{2\pi}} e^{im\phi} R(p). \quad (16)$$

Here, m is the quantum number associated to the operator L_z and $R(p)$ is the radial part of the wave function where in two dimensions is given by [16]

$$R_{n,m}^a(p) = \sqrt{\frac{2\beta(2n' + a + |m| + 1)n'!\Gamma(n' + a + |m| + 1)}{\Gamma(n' + a + 1)\Gamma(n' + |m| + 1)}} \beta^{|m|/2} (1 + \beta p^2)^{-(a+2+|m|)/2} p^{|m|} P_{n'}^{(a,|m|)}(z), \quad (17)$$

where $z = \frac{\beta p^2 - 1}{\beta p^2 + 1}$, $n' = (n - |m|)/2$, n is principal quantum numbers, $a = \sqrt{1 + m^2 + k^{-4}}$, $k = \sqrt{\mu\hbar\omega\beta}$, and $P_n^{(a,|m|)}(z)$ is the Jacobi polynomial. Also, its energy spectrum reads

$$E_{n,m} = \hbar\omega \left[(n+1)\sqrt{1 + \beta^2(m^2 + 1)\mu^2\hbar^2\omega^2} + \frac{\mu\hbar\omega\beta}{2} ((n+1)^2 + m^2 + 1) \right]. \quad (18)$$

4 The exact solution of Dirac equation with algebraic method

Using Eqs. (8), (10) and (13), the Hamiltonian for the first component of the spinor is given by

$$\begin{aligned} H^{(1)} &= P_- P_+ = P_x^2 + P_y^2 + \alpha^2(X^2 + Y^2) + 2\alpha(XP_y - YP_x) + i\alpha^2(XY - YX) - i\alpha[X, P_x] - i\alpha[Y, P_y], \\ &= (1 + 2\alpha\beta\hbar)P^2 + \alpha^2(X^2 + Y^2) + 2\alpha(1 + \beta P^2)L_z + 2\beta\alpha^2\hbar(1 + \beta P^2)L_z + 2\alpha\hbar, \end{aligned} \quad (19)$$

where $\alpha = eB_0/(2c)$. Using the ansatz

$$\psi^{(1)}(p) = \frac{1}{\sqrt{2\pi}} e^{im\phi} R^{(1)}(p), \quad (20)$$

that satisfies $L_z\psi^{(1)}(p) = m\hbar\psi^{(1)}(p)$ we obtain

$$H^{(1)}\psi^{(1)}(p) = \{[1 + 2\alpha\beta\hbar + 2\alpha\beta\hbar m(1 + \alpha\beta\hbar)]P^2 + \alpha^2(X^2 + Y^2) + 2\alpha\hbar m(1 + \alpha\beta\hbar) + 2\alpha\hbar\}\psi^{(1)}(p). \quad (21)$$

Now, this equation is similar to the Hamiltonian equation of the harmonic oscillator, namely $H^{(1)} =$

$\frac{1}{2\mu_1}P^2 + \frac{1}{2}\mu_1\omega_1^2(X^2 + Y^2) + c_1$, where $\mu_1 = [2 + 4\alpha\beta\hbar + 4\alpha\beta\hbar m(1 + \alpha\beta\hbar)]^{-1}$, $\omega_1^2 = 2\alpha^2[2 + 4\alpha\beta\hbar +$

$4\alpha\beta\hbar m(1+\alpha\beta\hbar)]$, and $c_1 = 2\alpha\hbar m(1+\alpha\beta\hbar) + 2\alpha\hbar$. So, the normalized radial part of energy eigenfunction for the first component is given by

$$R^{(1)}(p) = R_{n,m}^{a_1}(p), \quad (22)$$

where $a_1 = \sqrt{1 + m^2 + k_1^{-4}}$ and $k_1 = \sqrt{\mu_1\omega_1\hbar\beta}$. Also, the eigenvalues of the Hamiltonian $H^{(1)}$ are easily obtained

$$\epsilon_{n,m}^{2(1)} = 2\hbar\alpha(m+n+2) + \beta\hbar^2\alpha^2(m+n)^2 + 4\beta\hbar^2\alpha^2(m+n+1), \quad (23)$$

and the energy spectrum reads

$$E_{n,m} = \pm \sqrt{M^2c^4 + 2\hbar\alpha c^2(m+n+2)[1 + \frac{\hbar\alpha\beta}{2}(m+n+2)]}. \quad (24)$$

For the second component of the spinor, the Hamiltonian is

$$\begin{aligned} H^{(2)} &= P_+P_- = P_x^2 + P_y^2 + \alpha^2(X^2 + Y^2) + 2\alpha(XP_y - YP_x) - i\alpha^2(XY - YX) + i\alpha[X, P_x] + i\alpha[Y, P_y], \\ &= (1 - 2\alpha\beta\hbar)P^2 + \alpha^2(X^2 + Y^2) + 2\alpha(1 + \beta P^2)L_z - 2\beta\alpha^2\hbar(1 + \beta P^2)L_z - 2\alpha\hbar. \end{aligned} \quad (25)$$

Using the ansatz

$$\psi^{(2)}(p) = \frac{1}{\sqrt{2\pi}} e^{im'\phi} R^{(2)}(p), \quad (26)$$

and $L_z\psi^{(2)}(p) = m'\hbar\psi^{(2)}(p)$ we obtain

$$H^{(2)}\psi^{(2)}(p) = \{[1 - 2\alpha\beta\hbar + 2\alpha\beta(1 - \alpha\beta\hbar)m'\hbar]P^2 + \alpha^2(X^2 + Y^2) + 2\alpha(1 - \alpha\beta\hbar)m'\hbar - 2\alpha\hbar\}\psi^{(2)}(p). \quad (27)$$

This Hamiltonian is again similar to the harmonic oscillator Hamiltonian $H^{(2)} = \frac{1}{2\mu_2}P^2 + \frac{1}{2}\mu_2\omega_2^2(X^2 + Y^2) + c_2$, where $\mu_2 = [2 - 4\alpha\beta\hbar + 4\alpha\beta\hbar m'(1 - \alpha\beta\hbar)]^{-1}$, $\omega_2^2 = 2\alpha^2[2 - 4\alpha\beta\hbar + 4\alpha\beta\hbar m'(1 - \alpha\beta\hbar)]$, and $c_2 = 2\alpha(1 - \alpha\beta\hbar)m'\hbar - 2\alpha\hbar$. Therefore, the normalized radial part of the solution for the second component is given by

$$R^{(2)}(p) = R_{n,m'}^{a_2}(p), \quad (28)$$

where $a_2 = \sqrt{1 + m'^2 + k_2^{-4}}$, and $k_2 = \sqrt{\mu_2\omega_2\hbar\beta}$. Also, the eigenvalues of Eq. (27) read

$$\epsilon_{n,m'}^{2(2)} = \beta\hbar^2\alpha^2(m' + n)^2 + 2\hbar\alpha(m' + n), \quad (29)$$

and the energy spectrum can be written as

$$E_{n,m'} = \pm \sqrt{M^2 c^4 + 2\hbar\alpha c^2(m' + n)[1 + \frac{\hbar\alpha\beta}{2}(m' + n)]}. \quad (30)$$

Note that, since $\epsilon_{n,m}^{2(1)}$ should be equal to $\epsilon_{n,m'}^{2(2)}$, using Eqs. (23) and (29), we obtain $m' = m + 2$.

For a particular case, suppose that the quantum numbers take even values, i.e. $(n, m, m') \rightarrow (2n, 2m, 2m')$. For the first component we obtain

$$\epsilon_{n,m}^{2(1)} = 4\hbar\alpha(m + n + 1) + 4\beta\hbar^2\alpha^2(m + n)^2 + 8\beta\hbar^2\alpha^2(m + n) + 4\beta\hbar^2\alpha^2, \quad (31)$$

and

$$E_{n,m} = \pm \sqrt{M^2 c^4 + 4\hbar\alpha c^2(m + n + 1)[1 + \hbar\alpha\beta(m + n + 1)]}, \quad (32)$$

in agreement with Ref. [36]. Also, for the second component we have

$$\epsilon_{n,m'}^{2(2)} = 4\beta\hbar^2\alpha^2(m' + n)^2 + 4\hbar\alpha(m' + n), \quad (33)$$

and

$$E_{n,m'} = \pm \sqrt{M^2 c^4 + 4\hbar\alpha c^2(m' + n)[1 + \hbar\alpha\beta(m' + n)]}. \quad (34)$$

Now, the condition $\epsilon_{n,m}^{2(1)} = \epsilon_{n,m'}^{2(2)}$ implies $m' = m + 1$ that agrees with Ref. [36]. These results show that the set of solutions obtained by Menculini *et al.* is a subset of general solutions presented in this section which corresponds to the even quantum numbers.

For the massless (2+1)-dimensional Dirac equation that describes the motion of electrons in new materials such as graphene, the Hamiltonian is given by

$$H = v_F \vec{\sigma} \cdot \left(\vec{P} + \frac{e}{c} \vec{A} \right). \quad (35)$$

Here, v_F is the Fermi velocity which for electrons in graphene this velocity is more than the speed of light. For this Hamiltonian, the energy eigenvalues read

$$E_{n,m} = v_F \sqrt{2\hbar\alpha(m + n) \left[1 + \frac{\hbar\alpha\beta}{2}(m + n) \right]}. \quad (36)$$

Using this energy spectrum an upper bound on the fundamental minimal length can be estimated by comparison with the experimental results of the relativistic Landau levels in graphene [34]. For example, for $B_0 = 18 T$ and $v_F = (1.12 \pm 0.02) \times 10^6 \text{ m/s}$, the first experimental exited level of the graphene Landau spectrum in the absence of GUP is $E = (172 \pm 3) \text{ meV}$ [34]. So, by setting $n = 1$ and $m = 0$ in Eq. (36), we obtain

$$E_{1,0}^{(\beta)} = v_F \sqrt{2\hbar\alpha \left(1 + \frac{\hbar\alpha\beta}{2}\right)} = E_{1,0}^{(\beta=0)} \sqrt{1 + \frac{\hbar\alpha\beta}{2}}. \quad (37)$$

Now, since $\Delta E = E_{1,0}^{(\beta)} - E_{1,0}^{(\beta=0)} < 6 \text{ meV}$, we have

$$\Delta E = E_{1,0}^{(\beta=0)} \left(\sqrt{1 + \frac{\hbar\alpha\beta}{2}} - 1 \right) < 6 \text{ meV}. \quad (38)$$

If we define $\delta = \beta\hbar\alpha/2 = \alpha(\hbar\sqrt{\beta})^2/(2\hbar)$, we obtain $\delta < 0.07$. Therefore, the upper bound of the minimal length is found as

$$(\Delta X)_{min} = \hbar\sqrt{\beta} < 3.25 \text{ nm}, \quad (39)$$

which agrees with the ultracold neutron energy levels in a gravitational field in the presence of a minimal length [19] and a minimal length and a maximal momentum [35]. Note that the intermediate length scales obtained in Ref. [29] are $\ell_{\text{inter}} \sim 10^{10}\ell_{\text{Pl}}$, $10^{18}\ell_{\text{Pl}}$ and $10^{25}\ell_{\text{Pl}}$ for the potential barrier, Lamb shift, and Landau levels, respectively. Also, based on current experiments in superconductivity and muon experiments, the intermediate length scales are given by $\ell_{\text{inter}} \sim 10^{17}\ell_{\text{Pl}}$ and $10^8\ell_{\text{Pl}}$, respectively [30]. Thus, our result (39) also agrees with the upper bound predicted by the Landau levels [29]. Although our result is far weaker than the upper bound predicted by electroweak measurements, but it is not incompatible with it.

5 Conclusions

In this paper, we exactly solved the (2+1)-dimensional Dirac equation in a constant magnetic field in the framework of the generalized uncertainty principle. Using proper ansatzs for the momentum space wave functions, we transformed the Hamiltonian for each component of spinor into non-relativistic harmonic oscillator Hamiltonians. Then, the solutions are obtained without directly solving the GUP-corrected

Dirac equation. We also showed that Menculini *et al.* solutions correspond to the even quantum numbers as a subset of the general solution. For the massless case, we used the experimental results to find an upper bound for the deformation parameter which agreed with the ultracold neutron energy levels in a gravitational quantum well.

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